An open-boundary integrable model of three coupled XY spin chains

Anthony J. Bracken, Xiang-Yu Ge *, Yao-Zhong Zhang † and Huan-Qiang Zhou ‡

Department of Mathematics, University of Queensland, Brisbane, Qld 4072, Australia

The integrable open-boundary conditions for the model of three coupled one-dimensional XY spin chains are considered in the framework of the quantum inverse scattering method. The diagonal boundary K-matrices are found and a class of integrable boundary terms is determined. The boundary model Hamiltonian is solved by using the coordinate space Bethe ansatz technique and Bethe ansatz equations are derived.

I. INTRODUCTION

Many systems in nature have boundaries. Therefore the problem of how to extend a bulk integrable model to incorporate integrable open-boundary conditions becomes very important. In general, boundary conditions spoil the integrability of the bulk system and thus boundary models are not generally integrable.

One of the recent developments in the theory of completely integrable lattice systems with boundaries is Sklyanin's work [1] on the boundary quantum inverse scattering method (QISM). This work is extended by Mezinzescu et al [2] to treat boundary systems with the R-matrices satisfying the less restrictive condition of crossing-symmetry. It is noted by de Vega et al [3] that if R-matrices satisfy a weaker version of crossing-symmetry, called crossing-unitarity, the corresponding boundary systems can still be treated by the approach of Mezinzescu et al. Boundary integrable lattice systems have subsequently been extensively investigated in the literature (see, e.g. [4–9]).

In many interesting cases the R-matrices enjoy neither PT invariance and crossing-symmetry nor any kind of weaker versions such as crossing-unitarity. Two non-trivial examples are R-matrices [10] corresponding to the models of two coupled and three coupled one-dimensional XY spin chains introduced in [11]. For such systems, the formalism of [1,2] is not applicable. In [7] ([9]), a very general (graded) reflection equation (RE) algebra has been proposed and the corresponding (supersymmetric) boundary QISM has been formulated. The formalism applies to any (supersymmetric) lattice boundary system where an invertible R-matrix exists.

The aim of this paper is to study integrable open-boundary conditions for the model of three coupled one dimensional XY spin chains. The quantum integrability of the bulk model has been established in [10] by deriving the explicit form of the quantum R-matrix. It is seen that this R-matrix does not possess crossing-unitarity. So we apply the generalized scheme developed in [7,9], which is reformulated in section II. We are interested in constructing diagonal boundary K-matrices. Since the R-matrix for the model concerned is a 64×64 matrix with 216 non-zero elements, the whole procedure of solving REs for K-matrices is very involved. Furthermore, due to the absence of crossing-unitarity, there is no isomorphism between the two K-matrices $K_\pm$ and the two REs have to be solved separately. This fact is in contrast to the known cases in the literature, where K_+ can be derived from K_- by means of the isomorphism betweem them. In the present case, however, deriving K_+ is highly non-trivial, as it also involves determining the inverse of the partially transposed R-matrix. We list our results for the K-matrices in section III and more details can be found in the Appendices A and B. The explicit formulae for $K_\pm(u)$ verify that the two K-matrices are not related by automorphisms. In section IV we use these K-matrices to determine the open-boundary model Hamiltonian, which turns out to be a complicated algebraic manipulation. The reason is that at the zero spectral-parameter point, the traces of the K-matrix and of its 1st and 2nd derivatives are all equal to zero, and thus we have to expand the boundary transfer matrix to the 4-th order in the spectral parameter. The Hamiltonian of the boundary model is extracted from the 4-th derivative of the boundary transfer matrix. We then solve the boundary model by means of the coordinate Bethe ansatz method and derive the Bethe ansatz equations. In Appendix A, we sketch the procedure of solving the RE for $K_-(u)$, and in Appendix B we list the 216 non-zero matrix elements of $\tilde{R}(u)$ appearing in the RE for $K_+(u)$. The construction of $K_+(u)$ is then similar to that of $K_-(u)$. As a sideline, we show in Appendix C that all R-matrices associated with finite dimensional representations of a quantum affine algebra enjoy the crossing-unitarity property. We emphasize, however, that the R-matrix used in the present paper does not belong to this class.

*E-mail: xg@maths.uq.edu.au

†Queen Elizabeth II Fellow. E-mail: yzz@maths.uq.edu.au

‡On leave of absence from Dept of Physics, Chongqing University, Chongqing 630044, China. E-mail: hqzhou@cqu.edu.cn

II. REFLECTION EQUATIONS AND THE BOUNDARY TRANSFER MATRIX

Let V be a finite-dimensional linear space. Let the operator-valued function $R : C \to End(V \otimes V)$ be a solution to the quantum Yang-Baxter equation (QYBE)

$$R_{12}(u_1, u_2)R_{13}(u_1, u_3)R_{23}(u_2, u_3) = R_{23}(u_2, u_3)R_{13}(u_1, u_3)R_{12}(u_1, u_2). \tag{II.1}$$

Here $R_{jk}(u)$ denotes the matrix on $V \otimes V \otimes V$ acting on the j-th and k-th spaces and as an identity on the remaining space. The variables u_1, u_2 and u_3 are spectral parameters. No assumption that the spectral parameters are additive is to be made. Let P be the permutation operator in $V \otimes V$, i.e., $P(x \otimes y) = y \otimes x$, $\forall x, y \in V$. Then $R_{21}(u) = P_{12}R_{12}(u)P_{12}$. With the help of the R-matrix $R_{jk}(u)$, we form the monodromy matrix $T(u)$ for an L-site spin chain,

$$T(u) = L_{0L}(u) \cdots L_{01}(u) \tag{II.2}$$

where the subscript 0 labels the auxiliary space V and $L_{0j}(u) \equiv R_{0j}(u, 0)$. Indeed, one can show that $T(u)$ generates a representation of the quantum Yang-Baxter algebra,

$$R_{12}(u_1, u_2) \overset{1}{T} (u_1) \overset{2}{T} (u_2) = \overset{2}{T} (u_2) \overset{1}{T} (u_1)R_{12}(u_1, u_2). \tag{II.3}$$

where $\overset{1}{X} \equiv X \otimes 1$ and $\overset{2}{X} \equiv 1 \otimes X$, for any matrix $X \in End(V)$.

In order to construct open-boundary integrable spin chains, we introduce the following REs which the so-called boundary K-matrices satisfy:

$$\begin{aligned} R_{12}(u_1, u_2) \overset{1}{K}_- (u_1)R_{21}(u_2, -u_1) \overset{2}{K}_- (u_2) &= \overset{2}{K}_- (u_2)R_{12}(u_1, -u_2) \overset{1}{K}_- (u_1)R_{21}(-u_2, -u_1), \\ R_{21}^{t_1t_2}(u_2, u_1) \overset{1}{K}{}_+^{t_1} (u_1)\tilde{R}_{12}(-u_1, u_2) \overset{2}{K}{}_+^{t_2} (u_2) &= \overset{2}{K}{}_+^{t_2} (u_2)\tilde{R}_{21}(-u_2, u_1) \overset{1}{K}{}_+^{t_1} (u_1)R_{12}^{t_1t_2}(-u_1, -u_2), \end{aligned} \tag{II.4}$$

where we have defined a new object $\tilde{R}$ through the relation

$$\tilde{R}_{21}^{t_1}(-u_2, u_1)R_{12}^{t_2}(u_1, -u_2) = 1 \tag{II.5}$$

and t_i stands for the transposition taken in the i-th space. In all cases, quantum R-matrices enjoy the unitarity property [1],

$$R_{12}(u_1, u_2)R_{21}(u_2, u_1) = 1. \tag{II.6}$$

We now show that the second RE in (II.4) is indeed the correct "conjugation" to the first one, so that the boundary transfer matrices defined as usual constitute a commuting family. Following Sklyanin's arguments [1], one may show that the quantity $\mathcal{T}_-(u)$ given by

$$\mathcal{T}_-(u) = T(u)K_-(u)T^{-1}(-u) \tag{II.7}$$

satisfies the same relation as $K_-(u)$:

$$R_{12}(u_1, u_2) \overset{1}{\mathcal{T}}_- (u_1)R_{21}(u_2, -u_1) \overset{2}{\mathcal{T}}_- (u_2) = \overset{2}{\mathcal{T}}_- (u_2)R_{12}(u_1, -u_2) \overset{1}{\mathcal{T}}_- (u_1)R_{21}(-u_2, -u_1). \tag{II.8}$$

Thus if one defines the boundary transfer matrix as

$$t(u) = tr(K_+(u)\mathcal{T}_-(u)) = tr\left(K_+(u)T(u)K_-(u)T^{-1}(-u)\right), \tag{II.9}$$

where tr denotes the trace taken over the auxiliary space V, then it can be shown [9] that

$$[t(u_1), t(u_2)] = 0. \tag{II.10}$$

The REs (II.4) are generalizations of those introduced by Sklyanin [1] and Mezincescu et al [2]. In contrast to those works, we do not impose any constraint conditions on the R-matrix. Therefore the REs (II.4) apply to any bosonic (or non-supersymmetric) case where an invertible R-matrix exists.

[1] One can always normalize the R-matrices so that the right hand side of (II.6) is equal to the identity, as shown.

III. BOUNDARY K-MATRICES FOR THE COUPLED SPIN CHAIN MODEL

We consider a spin chain model defined by the following Hamiltonian

$$H = \sum_{j=1}^{L-1} H_{j,j+1} + B_L + B_R, \tag{III.1}$$

where $H_{j,j+1}$ denotes the bulk Hamiltonian density of three XY spin chains coupled to each other [11]

$$H_{j,j+1} = \sum_{\alpha} (\sigma^+_{j(\alpha)}\sigma^-_{j+1(\alpha)} + \sigma^-_{j(\alpha)}\sigma^+_{j+1(\alpha)}) \exp[\eta \sum_{\alpha'\neq\alpha} \sigma^+_{j+\theta(\alpha'-\alpha)(\alpha')}\sigma^-_{j+\theta(\alpha'-\alpha)(\alpha')}]. \tag{III.2}$$

Here $\sigma^{\pm}_{j(\alpha)} = \frac{1}{2}(\sigma^x_{j(\alpha)} \pm i\sigma^y_{j(\alpha)})$, with $\sigma^x_{j(\alpha)}, \sigma^y_{j(\alpha)}, \sigma^z_{j(\alpha)}$ being the usual Pauli spin operators at site j corresponding to the α-th ($\alpha = 1, 2, 3$) XY spin chain; $\theta(\alpha' - \alpha)$ is a step function of $(\alpha' - \alpha)$; and η is a coupling constant. The left and right boundary terms B_L and B_R have the form

$$\begin{aligned}
B_L &= \frac{1}{2c_- \exp(2\eta)}[\cosh^2\eta \sum_{\alpha}\sigma^z_{1(\alpha)} + \frac{\sinh\eta\cosh\eta}{2}\sum_{\alpha\neq\beta}\sigma^z_{1(\alpha)}\sigma^z_{1(\beta)} + \sinh^2\eta\, \sigma^z_{1(1)}\sigma^z_{1(2)}\sigma^z_{1(3)}],\\
B_R &= \frac{c_+ \exp(6\eta)}{2}[\cosh^2\eta \sum_{\alpha}\sigma^z_{L(\alpha)} + \frac{\sinh\eta\cosh\eta}{2}\sum_{\alpha\neq\beta}\sigma^z_{L(\alpha)}\sigma^z_{L(\beta)} + \sinh^2\eta\, \sigma^z_{L(1)}\sigma^z_{L(2)}\sigma^z_{L(3)}],
\end{aligned} \tag{III.3}$$

where $c_\pm$ are parameters describing the boundary effects. After a generalized Jordan-Wigner transformation, the Hamiltonian (III.1) becomes a strongly correlated electronic system with boundary interactions. If one restricts the Hilbert space to the one which only consists of, say, $\sigma_{(1)}$, $\sigma_{(2)}$, then Hamiltonian (III.1) reduces to that of two coupled XY open chains with special boundary interactions, which has been considered in [7].

We shall establish the quantum integrability of the system defined by the Hamiltonian (III.1), by using the general formalism described in the previous section. Let us first of all recall some basic results for the bulk model (III.2) with the periodic boundary conditions. As was shown in [10], the bulk model Hamiltonian commutes with a one-parameter family of bulk transfer matrices $\tau(u)$ of a two-dimensional lattice statistical mechanics model. This transfer matrix is the trace of a monodromy matrix $T(u)$ with $L_{0j}(u)$ of the form,

$$L_{0j}(u) = L^{(1)}_{0j}(u)L^{(2)}_{0j}(u)L^{(3)}_{0j}(u), \tag{III.4}$$

where

$$\begin{aligned}
L^{(\alpha)}_{0j}(u) &= \frac{1}{2}(1 + \sigma^z_{j(\alpha)}\sigma^z_{0(\alpha)}) + \frac{1}{2}u(1 - \sigma^z_{j(\alpha)}\sigma^z_{0(\alpha)}) \exp(\eta \sum_{\substack{\alpha'=1\\ \alpha'\neq\alpha}}^{3} \sigma^+_{0(\alpha')}\sigma^-_{0(\alpha')})\\
&\quad +(\sigma^-_{j(\alpha)}\sigma^+_{0(\alpha)} + \sigma^+_{j(\alpha)}\sigma^-_{0(\alpha)})\sqrt{1 + \exp(2\eta \sum_{\substack{\alpha'=1\\ \alpha'\neq\alpha}}^{3} \sigma^+_{0(\alpha')}\sigma^-_{0(\alpha')})u^2}.
\end{aligned} \tag{III.5}$$

The commutativity of the bulk transfer matrices $\tau(u)$ for different values of the spectral parameter u follows from the fact that the L matrix $L_{0j}(u)$ (III.4) satisfies the Yang-Baxter algebra (II.3). The explicit form of the corresponding R-matrix $R_{12}(u_1, u_2)$ can be found in the third reference in [10]. Here we only emphasize that the local monodromy matrix as well as the quantum R-matrix does not possess crossing symmetry and crossing-unitarity.

In order to describe integrable systems with boundary conditions different from the periodic ones, let us first of all solve the REs for the two boundary K-matrices $K_\pm(u)$. As mentioned in the Introduction, there is no isomorphism between $K_-(u)$ and $K_+(u)$, and thus we have to solve the two REs separately. For our purpose, we only look for solutions where $K_\pm(u)$ are diagonal. After complicated algebraic manipulations (for a sketch of the whole procedure, see Appendix A), we find

$$K_-(u) = \frac{1}{\exp(6\eta)c_-^3}\begin{pmatrix} A_-(u) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B_-(u) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & B_-(u) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_-(u) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_-(u) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & C_-(u) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & C_-(u) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & D_-(u) \end{pmatrix}, \tag{III.6}$$

with

$$\begin{aligned} A_-(u) &= (c_- + u)(e^{2\eta}c_- + u)(e^{4\eta}c_- + u), \\ B_-(u) &= (c_- - u)(e^{2\eta}c_- + u)(e^{4\eta}c_- + u), \\ C_-(u) &= (c_- - u)(e^{2\eta}c_- - u)(e^{4\eta}c_- + u), \\ D_-(u) &= (c_- - u)(e^{2\eta}c_- - u)(e^{4\eta}c_- - u). \end{aligned}$$

It is much more tedious to find the boundary K-matrix $K_+(u)$, since the corresponding RE is more involved. We list the final result here and some details can be found in Appendices B and A,

$$K_+(u) = \begin{pmatrix} A_+(u) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B_+(u) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C_+(u) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & D_+(u) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & E_+(u) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & F_+(u) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & G_+(u) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & H_+(u) \end{pmatrix}, \tag{III.7}$$

with

$$\begin{aligned} A_+(u) &= e^{6\eta}(c_+ + u)(e^{2\eta}c_+ + u)(e^{4\eta}c_+ + u), \\ B_+(u) &= e^{6\eta}(-e^{4\eta}c_+ + u)(e^{2\eta}c_+ + u)(e^{4\eta}c_+ + u), \\ C_+(u) &= e^{4\eta}(c_+ + u)(-e^{4\eta}c_+ + u)(e^{4\eta}c_+ + u), \\ D_+(u) &= e^{4\eta}(-e^{4\eta}c_+ + u)(-e^{6\eta}c_+ + u)(e^{4\eta}c_+ + u), \\ E_+(u) &= e^{2\eta}(e^{2\eta}c_+ + u)(-e^{4\eta}c_+ + u)(e^{4\eta}c_+ + u), \\ F_+(u) &= e^{2\eta}(-e^{4\eta}c_+ + u)(-e^{6\eta}c_+ + u)(e^{4\eta}c_+ + u), \\ G_+(u) &= (-e^{4\eta}c_+ + u)(-e^{6\eta}c_+ + u)(e^{4\eta}c_+ + u), \\ H_+(u) &= (-e^{4\eta}c_+ + u)(-e^{6\eta}c_+ + u)(-e^{8\eta}c_+ + u). \end{aligned}$$

The above explicit formulae for $K_\pm(u)$, derived by solving the two REs separately, show that no automorphism between $K_+(u)$ and $K_-(u)$ exists and $K_+(u)$ cannot be obtained from $K_-(u)$, as claimed.

IV. EMBEDDING OF THE HAMILTONIAN INTO THE BOUNDARY TRANSFER MATRIX AND THE BETHE ANSATZ EQUATIONS

To show that the Hamiltonian (III.1) can be embedded into the boundary transfer matrix $t(u)$ constructed in section II, is an involved algebraic manipulation. This is because the traces of $K_+(0)$, and of its first derivative $K'_+(0)$ and second derivative $K''_+(0)$, are all equal to zero. So at best we can only expect that the Hamiltonian (III.1) appears as the fourth derivative of the boundary transfer matrix with respect to the spectral parameter u, at $u = 0$.

Let us expand the local monodromy matrix $L_{0j}(u)$ to the fourth order in the spectral parameter u,

$$L_{0j}(u) = (1 + H_{j0}u + \frac{1}{2!}B_{j0}u^2 + \frac{1}{3!}C_{j0}u^3 + \frac{1}{4!}D_{j0}u^4 + \cdots)L_{0j}(0), \tag{IV.1}$$

where,

$$
\begin{aligned}
H_{j0} &= H^{(1)}_{j0} + P^{(1)}_{j0} H^{(2)}_{j0} P^{(1)}_{j0} + P^{(1)}_{j0} P^{(2)}_{j0} H^{(3)}_{j0} P^{(2)}_{j0} P^{(1)}_{j0},\\
B_{j0} &= B^{(1)}_{j0} + P^{(1)}_{j0} B^{(2)}_{j0} P^{(1)}_{j0} + P^{(1)}_{j0} P^{(2)}_{j0} B^{(3)}_{j0} P^{(2)}_{j0} P^{(1)}_{j0} + 2H^{(1)}_{j0} P^{(1)}_{j0} H^{(2)}_{j0} P^{(1)}_{j0}\\
&\quad +2H^{(1)}_{j0} P^{(1)}_{j0} P^{(2)}_{j0} H^{(3)}_{j0} P^{(2)}_{j0} P^{(1)}_{j0} + 2P^{(1)}_{j0} H^{(2)}_{j0} P^{(1)}_{j0} P^{(1)}_{j0} P^{(2)}_{j0} H^{(3)}_{j0} P^{(2)}_{j0} P^{(1)}_{j0},\\
C_{j0} &= 3B^{(1)}_{j0} P^{(1)}_{j0} H^{(2)}_{j0} P^{(1)}_{j0} + 3B^{(1)}_{j0} P^{(1)}_{j0} P^{(2)}_{j0} H^{(3)}_{j0} P^{(2)}_{j0} P^{(1)}_{j0}\\
&\quad +3H^{(1)}_{j0} P^{(1)}_{j0} B^{(2)}_{j0} P^{(1)}_{j0} + 3H^{(1)}_{j0} P^{(1)}_{j0} P^{(2)}_{j0} B^{(3)}_{j0} P^{(2)}_{j0} P^{(1)}_{j0}\\
&\quad +3P^{(1)}_{j0} B^{(2)}_{j0} P^{(1)}_{j0} P^{(1)}_{j0} P^{(2)}_{j0} H^{(3)}_{j0} P^{(2)}_{j0} P^{(1)}_{j0} + 3P^{(1)}_{j0} H^{(2)}_{j0} P^{(1)}_{j0} P^{(1)}_{j0} P^{(2)}_{j0} B^{(3)}_{j0} P^{(2)}_{j0} P^{(1)}_{j0}\\
&\quad +6H^{(1)}_{j0} P^{(1)}_{j0} H^{(2)}_{j0} P^{(1)}_{j0} P^{(1)}_{j0} P^{(2)}_{j0} H^{(3)}_{j0} P^{(2)}_{j0} P^{(1)}_{j0}
\end{aligned} \tag{IV.2}
$$

with

$$
\begin{aligned}
H^{(\alpha)}_{j0} &= (\sigma^-_{j(\alpha)}\sigma^+_{0(\alpha)} + \sigma^+_{j(\alpha)}\sigma^-_{0(\alpha)}) \exp(\eta \sum_{\substack{\alpha'=1\\ \alpha\neq\alpha'}}^{3} \sigma^+_{0(\alpha')}\sigma^-_{0(\alpha')}),\\
P^{(\alpha)}_{j0} &= \frac{1+\sigma^z_{j(\alpha)}\sigma^z_{0(\alpha)}}{2} + (\sigma^-_{j(\alpha)}\sigma^+_{0(\alpha)} + \sigma^+_{j(\alpha)}\sigma^-_{0(\alpha)}),\\
B^{(\alpha)}_{j0} &= (\sigma^-_{j(\alpha)}\sigma^+_{0(\alpha)} + \sigma^+_{j(\alpha)}\sigma^-_{0(\alpha)}) P^{(\alpha)}_{j0} \exp(2\eta \sum_{\substack{\alpha'=1\\ \alpha\neq\alpha'}}^{3} \sigma^+_{0(\alpha')}\sigma^-_{0(\alpha')}).
\end{aligned} \tag{IV.3}
$$

Substituting the expansion for $L_{0j}(u)$ into the boundary transfer matrix $t(u)$, and after a lengthy but straightforward algebraic calculation, one finds

$$
t(u) = C_1 u^3 + C_2 (H + const.) u^4 + \cdots, \tag{IV.4}
$$

where $C_i (i = 1, 2, \cdots)$ are some scalar functions of the boundary constant c_+. It can be shown that the Hamiltonian (III.1) is related to the fourth derivative of the boundary transfer matrix $t(u)$,

$$
H \equiv \frac{t^{(4)}(0)}{8\, trK'''_+(0)} = \sum_{j=1}^{L-1} H_{j,j+1} + \frac{1}{2} \overset{1}{K}{}'_-(0) + \frac{3}{trK'''_+(0)}\, tr(K''_+(0) H^2_{L0}), \tag{IV.5}
$$

where

$$
H_{j,j+1} = L_{0,j+1}(0) L'_{0j}(0) L^{-1}_{0j}(0) L^{-1}_{0,j+1}(0). \tag{IV.6}
$$

In deriving (IV.5) we have used the following relations which hold in the present case,

$$
\begin{aligned}
& trK^{(n)}_+(0) H_{L0} = 0, \qquad n = 0,1,2,3,\\
& trK^{(n)}_+(0) B_{L0} = 0, \qquad n = 0,1,\\
& trK_+(0) H_{L0} B_{L0} = 0, \qquad trK_+(0) B_{L0} H_{L0} = 0, \qquad tr\,(K_+(0) C_{L0} H_{L0}) = 0,\\
& trK_+(0) H^2_{L0} = 0, \qquad trK'_+(0) H^2_{L0} = 0, \qquad trK'_+(0) C_{L0} = 0,\\
& trK_+(0) H^3_{L0} = 0, \qquad trK'_+(0) H^3_{L0} = 0,\\
& tr\,(K_+(0) H_{L0} C_{L0}) = 0, \qquad tr\,(K_+(0) H_{L0} B_{L0} H_{L0}) = 0,\\
& tr\left(K_+(0) H^2_{L0} B_{L0}\right) = 0, \qquad tr\left(K_+(0) H^4_{L0}\right) = 0.
\end{aligned} \tag{IV.7}
$$

We have shown that the Hamiltonian (III.1) of the boundary model of three coupled one dimensional XY spin chains is related to a class of commuting transfer matrices. As a result, the system has an infinite number of higher conserved currents which are involutive with each other, and therefore the system under study is completely integrable.

Having established the quantum integrability of the model, let us now solve it by using the coordinate space Bethe ansatz method. The procedure is similar to that for other models [6,8,9]. The Bethe ansatz equations are

$$e^{ik_j 2(L+1)}\zeta(k_j;p_1)\zeta(k_j;p_L) = \prod_{\alpha=1}^{M_1} \frac{\sin[\frac{1}{2}(k_j-\Lambda^{(1)}_\alpha)+\frac{i\eta}{2}]}{\sin[\frac{1}{2}(k_j-\Lambda^{(1)}_\alpha)-\frac{i\eta}{2}]} \frac{\sin[\frac{1}{2}(k_j+\Lambda^{(1)}_\alpha)+\frac{i\eta}{2}]}{\sin[\frac{1}{2}(k_j+\Lambda^{(1)}_\alpha)-\frac{i\eta}{2}]},$$

$$\prod_{\alpha=1}^{N} \frac{\sin[\frac{1}{2}(\Lambda^{(1)}_\gamma-k_\alpha)+\frac{i\eta}{2}]}{\sin[\frac{1}{2}(\Lambda^{(1)}_\gamma-k_\alpha)-\frac{i\eta}{2}]} \frac{\sin[\frac{1}{2}(\Lambda^{(1)}_\gamma+k_\alpha+\frac{i\eta}{2}]}{\sin[\frac{1}{2}(\Lambda^{(1)}_\gamma+k_\alpha)-\frac{i\eta}{2}]} = \prod_{\substack{\gamma'=1\\ \gamma'\neq\gamma}}^{M_1} \frac{\sin[\frac{1}{2}(\Lambda^{(1)}_\gamma-\Lambda^{(1)}_{\gamma'})+i\eta]}{\sin[\frac{1}{2}(\Lambda^{(1)}_\gamma-\Lambda^{(1)}_{\gamma'})-i\eta]} \frac{\sin[\frac{1}{2}(\Lambda^{(1)}_\gamma+\Lambda^{(1)}_{\gamma'})+i\eta]}{\sin[\frac{1}{2}(\Lambda^{(1)}_\gamma+\Lambda^{(1)}_{\gamma'})-i\eta]}$$

$$\times \prod_{\delta=1}^{M_2} \frac{\sin[\frac{1}{2}(\Lambda^{(1)}_\gamma-\lambda^{(2)}_\delta)-\frac{i\eta}{2}]}{\sin[\frac{1}{2}(\Lambda^{(1)}_\gamma-\lambda^{(2)}_\delta)+\frac{i\eta}{2}]} \frac{\sin[\frac{1}{2}(\Lambda^{(1)}_\gamma+\lambda^{(2)}_\delta)-\frac{i\eta}{2}]}{\sin[\frac{1}{2}(\Lambda^{(1)}_\gamma+\lambda^{(2)}_\delta)+\frac{i\eta}{2}]},$$

$$\prod_{\substack{\gamma'=1\\ \gamma'\neq\gamma}}^{M_2} \frac{\sin[\frac{1}{2}(\Lambda^{(2)}_\gamma-\Lambda^{(2)}_{\gamma'})+i\eta]}{\sin[\frac{1}{2}(\Lambda^{(2)}_\gamma-\Lambda^{(2)}_{\gamma'})-i\eta]} \frac{\sin[\frac{1}{2}(\Lambda^{(2)}_\gamma+\Lambda^{(2)}_{\gamma'})+i\eta]}{\sin[\frac{1}{2}(\Lambda^{(2)}_\gamma+\Lambda^{(2)}_{\gamma'})-i\eta]} = \prod_{\alpha=1}^{M_1} \frac{\sin[\frac{1}{2}(\Lambda^{(2)}_\gamma-\Lambda^{(1)}_\alpha)+\frac{i\eta}{2}]}{\sin[\frac{1}{2}(\Lambda^{(2)}_\gamma-\Lambda^{(1)}_\alpha)-\frac{i\eta}{2}]} \frac{\sin[\frac{1}{2}(\Lambda^{(2)}_\gamma+\Lambda^{(1)}_\alpha)+\frac{i\eta}{2}]}{\sin[\frac{1}{2}(\Lambda^{(2)}_\gamma+\Lambda^{(1)}_\alpha)-\frac{i\eta}{2}]}, \quad \text{(IV.8)}$$

where

$$\zeta(k;p) = (1-pe^{-ik})/(1-pe^{ik}), \qquad p_1 = \frac{1}{c_- \exp(4\eta)}, \qquad p_L = c_+ \exp(4\eta). \quad \text{(IV.9)}$$

The energy eigenvalue E of the model is given by $E = 2\sum_{j=1}^{N}\cos k_j$ (modulo an unimportant additive constant, which we drop).

V. CONCLUSION

In this paper, we have studied integrable open-boundary conditions for the coupled spin chain model. The quantum integrability of the boundary system has been established by the fact that the corresponding Hamiltonian may be embedded into a one-parameter family of commuting transfer matrices. Moreover, the Bethe Ansatz equations are derived by means of the coordinate space Bethe ansatz approach. This provides us with a basis for computing the finite size corrections to the low-lying energies in the system, which in turn will allow us to use the boundary conformal field theory technique to study the critical properties of the boundary.

Let us add a few comments. Firstly, it should be emphasized that the formalism exposed here applies to all bosonic lattice systems where an invertible R-matrix exists, since one does not impose any constraint conditions on the quantum R-matrix. Secondly, the results can be generalized to the case of s coupled XY chains, with s arbitrary. It is interesting to note that in the case $s=2$, the Hamiltonian appears as the third derivative of the transfer matrix with respect to the spectral parameter u [7], whereas in the present case, $s=3$, it appears as the fouth derivative. Therefore, one may expect that for arbitrary s, the corresponding Hamiltonian will appear as the $(s+1)$-th derivative of the transfer matrix. Finally, it seems interesting to derive the Bethe ansatz equations by using the algebraic Bethe ansatz approach.

Y.-Z.Z and H.-Q.Z are supported by Australian Research Council, University of Queensland New Staff Research Grant and External Support Enabling Grant. H.-Q.Z would like to thank the Department of Mathematics, University of Queensland for kind hospitality. He is also partially supported by the National Natural Science Foundation of China and Sichuan Young Investigators Science and Technology Fund.

APPENDIX A: DERIVATION OF THE BOUNDARY K-MATRICES

In this appendix, we sketch the procedure of solving the RE for $K_-(u)$. The R-matrix appeared in this paper is an 64×64 matrix with 216 non-zero elements (see the third reference in [10]). We only consider diagonal solution. We parametrize $K_-(u)$ as

$$K_-(u) = \begin{pmatrix} a(u) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b(u) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c(u) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d(u) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e(u) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & f(u) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & g(u) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & h(u) \end{pmatrix}. \tag{A.1}$$

The RE for $K_-(u)$ consists of 216 functional equations, many of which are dependent. To solve them, we group these 216 functional equations into four categories: Category 0 contains 8 identities, each made of two terms; Category 1 contains 48 equations, each of 4 terms; Category 2 contains 96 equations, each of 8 terms; Category 3 contains 64 equations, each of 16 terms. Now divide the 48 functional equations in Category 1 into 12 sets of equations, so that each set contains 4 equations. It turns out that in any set of the 4 equations, two of them are satisfied identically and the other two are equivalent to each other. Therefore in these 48 equations in Category 1, only 12 need to be solved. Picking the following 7 equations out of those 12 equations,

$$\begin{aligned}
&a(u_1)a(u_2)R_{21,12}(u_1,u_2)R_{12,12}(u_2,-u_1) + b(u_1)a(u_2)R_{21,21}(u_1,u_2)R_{21,12}(u_2,-u_1) \\
&\quad = a(u_1)b(u_2)R_{21,12}(u_1,-u_2)R_{12,12}(-u_2,-u_1) + b(u_1)b(u_2)R_{21,21}(u_1,-u_2)R_{21,12}(-u_2,-u_1), && \text{(A.2)} \\
&a(u_1)a(u_2)R_{31,13}(u_1,u_2)R_{13,13}(u_2,-u_1) + c(u_1)a(u_2)R_{31,31}(u_1,u_2)R_{31,13}(u_2,-u_1) \\
&\quad = a(u_1)c(u_2)R_{31,13}(u_1,-u_2)R_{13,13}(-u_2,-u_1) + c(u_1)c(u_2)R_{31,31}(u_1,-u_2)R_{31,13}(-u_2,-u_1), && \text{(A.3)} \\
&a(u_1)a(u_2)R_{51,15}(u_1,u_2)R_{15,15}(u_2,-u_1) + e(u_1)a(u_2)R_{51,51}(u_1,u_2)R_{51,15}(u_2,-u_1) \\
&\quad = a(u_1)e(u_2)R_{51,15}(u_1,-u_2)R_{15,15}(-u_2,-u_1) + e(u_1)e(u_2)R_{51,51}(u_1,-u_2)R_{51,15}(-u_2,-u_1), && \text{(A.4)} \\
&b(u_1)b(u_2)R_{42,24}(u_1,u_2)R_{24,24}(u_2,-u_1) + d(u_1)b(u_2)R_{42,42}(u_1,u_2)R_{42,24}(u_2,-u_1) \\
&\quad = b(u_1)d(u_2)R_{42,24}(u_1,-u_2)R_{24,24}(-u_2,-u_1) + d(u_1)d(u_2)R_{42,42}(u_1,-u_2)R_{42,24}(-u_2,-u_1), && \text{(A.5)} \\
&b(u_1)b(u_2)R_{62,26}(u_1,u_2)R_{26,26}(u_2,-u_1) + f(u_1)b(u_2)R_{62,62}(u_1,u_2)R_{62,26}(u_2,-u_1) \\
&\quad = b(u_1)f(u_2)R_{62,26}(u_1,-u_2)R_{26,26}(-u_2,-u_1) + f(u_1)f(u_2)R_{62,62}(u_1,-u_2)R_{62,26}(-u_2,-u_1), && \text{(A.6)} \\
&c(u_1)c(u_2)R_{73,37}(u_1,u_2)R_{37,37}(u_2,-u_1) + g(u_1)c(u_2)R_{73,73}(u_1,u_2)R_{73,37}(u_2,-u_1) \\
&\quad = c(u_1)g(u_2)R_{73,37}(u_1,-u_2)R_{37,37}(-u_2,-u_1) + g(u_1)g(u_2)R_{73,73}(u_1,-u_2)R_{73,37}(-u_2,-u_1), && \text{(A.7)} \\
&d(u_1)d(u_2)R_{84,48}(u_1,u_2)R_{48,48}(u_2,-u_1) + h(u_1)d(u_2)R_{84,84}(u_1,u_2)R_{84,48}(u_2,-u_1) \\
&\quad = d(u_1)h(u_2)R_{84,48}(u_1,-u_2)R_{48,48}(-u_2,-u_1) + h(u_1)h(u_2)R_{84,84}(u_1,-u_2)R_{84,48}(-u_2,-u_1), && \text{(A.8)}
\end{aligned}$$

and using some nontrivial tricks of variable separation, we determine 7 constants, $c_1,\ c_2,\ \cdots,\ c_7$, by the relations

$$\begin{aligned}
&\frac{b(u)}{a(u)} = \frac{c_1-u}{c_1+u}, \qquad \frac{c(u)}{a(u)} = \frac{c_2-u}{c_2+u}, \qquad \frac{e(u)}{a(u)} = \frac{c_3-u}{c_3+u}, \\
&\frac{d(u)}{a(u)} = \frac{c_1-u}{c_1+u}\cdot\frac{c_4-u}{c_4+u}, \qquad \frac{f(u)}{a(u)} = \frac{c_1-u}{c_1+u}\cdot\frac{c_5-u}{c_5+u}, \\
&\frac{g(u)}{a(u)} = \frac{c_2-u}{c_2+u}\cdot\frac{c_6-u}{c_6+u}, \qquad \frac{h(u)}{a(u)} = \frac{c_1-u}{c_1+u}\cdot\frac{c_4-u}{c_4+u}\cdot\frac{c_7-u}{c_7+u}. && \text{(A.9)}
\end{aligned}$$

Then divide the 96 equations in Category 2 into 6 sets of equations. Each set contains 16 equations. In any set of the 16 equations, eight equations are identities and only two of the remaining 8 equations are relevant. Therefore out of these 96 equations in Category 2, only 12 equations can provide useful information. Inserting (A.9) into the following 4 equations picked from those 12 equations,

$$\begin{aligned}
&a(u_1)a(u_2)R_{41,14}(u_1,u_2)R_{14,14}(u_2,-u_1) + b(u_1)a(u_2)R_{41,23}(u_1,u_2)R_{23,14}(u_2,-u_1) \\
&\quad + c(u_1)a(u_2)R_{41,32}(u_1,u_2)R_{32,14}(u_2,-u_1) + d(u_1)a(u_2)R_{41,41}(u_1,u_2)R_{41,14}(u_2,-u_1) \\
&= a(u_1)d(u_2)R_{41,14}(u_1,-u_2)R_{14,14}(-u_2,-u_1) + b(u_1)d(u_2)R_{41,23}(u_1,-u_2)R_{23,14}(-u_2,-u_1) \\
&\quad + c(u_1)d(u_2)R_{41,32}(u_1,-u_2)R_{32,14}(-u_2,-u_1) + d(u_1)d(u_2)R_{41,41}(u_1,-u_2)R_{41,14}(-u_2,-u_1), && \text{(A.10)} \\
&a(u_1)a(u_2)R_{61,16}(u_1,u_2)R_{16,16}(u_2,-u_1) + b(u_1)a(u_2)R_{61,25}(u_1,u_2)R_{25,16}(u_2,-u_1) \\
&\quad + e(u_1)a(u_2)R_{61,52}(u_1,u_2)R_{52,16}(u_2,-u_1) + f(u_1)a(u_2)R_{61,61}(u_1,u_2)R_{61,16}(u_2,-u_1) \\
&= a(u_1)f(u_2)R_{61,16}(u_1,-u_2)R_{16,16}(-u_2,-u_1) + b(u_1)f(u_2)R_{61,25}(u_1,-u_2)R_{25,16}(-u_2,-u_1)
\end{aligned}$$

$$
\begin{aligned}
&\quad + e(u_1)f(u_2)R_{61,52}(u_1,-u_2)R_{52,16}(-u_2,-u_1) + f(u_1)f(u_2)R_{61,61}(u_1,-u_2)R_{61,16}(-u_2,-u_1), && \text{(A.11)}\\
&a(u_1)a(u_2)R_{71,17}(u_1,u_2)R_{17,17}(u_2,-u_1) + c(u_1)a(u_2)R_{71,35}(u_1,u_2)R_{35,17}(u_2,-u_1)\\
&\quad + e(u_1)a(u_2)R_{71,53}(u_1,u_2)R_{53,17}(u_2,-u_1) + g(u_1)a(u_2)R_{71,71}(u_1,u_2)R_{71,17}(u_2,-u_1)\\
&= a(u_1)g(u_2)R_{71,17}(u_1,-u_2)R_{17,17}(-u_2,-u_1) + c(u_1)g(u_2)R_{71,35}(u_1,-u_2)R_{35,17}(-u_2,-u_1)\\
&\quad + e(u_1)g(u_2)R_{71,53}(u_1,-u_2)R_{53,17}(-u_2,-u_1) + g(u_1)g(u_2)R_{71,71}(u_1,-u_2)R_{71,17}(-u_2,-u_1), && \text{(A.12)}\\
&b(u_1)b(u_2)R_{82,28}(u_1,u_2)R_{28,28}(u_2,-u_1) + d(u_1)b(u_2)R_{82,46}(u_1,u_2)R_{46,28}(u_2,-u_1)\\
&\quad + f(u_1)b(u_2)R_{82,64}(u_1,u_2)R_{64,28}(u_2,-u_1) + h(u_1)b(u_2)R_{82,82}(u_1,u_2)R_{82,28}(u_2,-u_1)\\
&= b(u_1)h(u_2)R_{82,28}(u_1,-u_2)R_{28,28}(-u_2,-u_1) + d(u_1)h(u_2)R_{82,46}(u_1,-u_2)R_{46,28}(-u_2,-u_1)\\
&\quad + f(u_1)h(u_2)R_{82,64}(u_1,-u_2)R_{64,28}(-u_2,-u_1) + h(u_1)h(u_2)R_{82,82}(u_1,-u_2)R_{82,28}(-u_2,-u_1), && \text{(A.13)}
\end{aligned}
$$

one can determine the following relationships among the 7 constants:

$$
c_2 = c_3 = c_1, \quad c_4 = c_5 = c_6 = e^{2\eta}c_1, \quad c_7 = e^{4\eta}c_1. \tag{A.14}
$$

So we have uniquely determined the ratios $\frac{b(u)}{a(u)}$ *etc.* up to a free parameter $c_1 \equiv c_-$. Finally it can be checked that the 64 functional equations in Category 3 are identically satisfied with the $\frac{b(u)}{a(u)}$ *etc.* determined as above. So we end up with the diagonal solution $K_-(u)$ in section III with one arbitrary parameter satisfying all 216 functional equations.

APPENDIX B: THE MATRIX ELEMENTS OF $\tilde{R}$

The first step towards solving the RE for $K_+(u)$ is to find $\tilde{R}(u,v)$ defined by (II.5). After long algebraic computations, one finds that the non-zero matrix elements of $\tilde{R}(u,v)$ are

$$
\begin{aligned}
\tilde{R}_{11,11}(u,v) &= -\frac{(1+uv)(1+e^{2\eta}uv)(1+e^{4\eta}uv)^2(1+e^{6\eta}uv)(1+e^{8\eta}uv)}{e^{6\eta}(u-v)^2(e^{2\eta}u-v)(u-e^{2\eta}v)(e^{4\eta}u-v)(u-e^{4\eta}v)},\\
\tilde{R}_{12,12}(u,v) &= \tilde{R}_{13,13}(u,v) = \tilde{R}_{15,15}(u,v) = \frac{(1+uv)(1+e^{2\eta}uv)(1+e^{4\eta}uv)^2(1+e^{6\eta}uv)}{e^{4\eta}(u-v)^2(e^{2\eta}u-v)(u-e^{2\eta}v)(u-e^{4\eta}v)},\\
\tilde{R}_{12,21}(u,v) &= \tilde{R}_{51,15}(u,v) = e^{2\eta}\tilde{R}_{13,31}(u,v) = e^{2\eta}\tilde{R}_{31,13}(u,v) = e^{4\eta}\tilde{R}_{15,51}(u,v) = e^{4\eta}\tilde{R}_{21,12}(u,v)\\
&= \frac{(1+uv)(1+e^{2\eta}uv)(1+e^{4\eta}uv)^2(1+e^{6\eta}uv)\sqrt{(1+e^{4\eta}u^2)(1+e^{4\eta}v^2)}}{e^{2\eta}(u-v)^2(e^{2\eta}u-v)(u-e^{2\eta}v)(e^{4\eta}u-v)(u-e^{4\eta}v)},\\
\tilde{R}_{14,14}(u,v) &= \tilde{R}_{16,16}(u,v) = \tilde{R}_{17,17}(u,v) = -\frac{(1+uv)(1+e^{2\eta}uv)(1+e^{4\eta}uv)^2}{e^{2\eta}(u-v)^2(u-e^{2\eta}v)(u-e^{4\eta}v)},\\
\tilde{R}_{14,23}(u,v) &= \tilde{R}_{16,25}(u,v) = \tilde{R}_{52,16}(u,v) = \tilde{R}_{53,17}(u,v)\\
&= e^{\eta}\tilde{R}_{14,32}(u,v) = e^{\eta}\tilde{R}_{35,17}(u,v) = e^{2\eta}\tilde{R}_{17,35}(u,v) = e^{2\eta}\tilde{R}_{32,14}(u,v)\\
&= e^{3\eta}\tilde{R}_{16,52}(u,v) = e^{3\eta}\tilde{R}_{17,53}(u,v) = e^{3\eta}\tilde{R}_{23,14}(u,v) = e^{3\eta}\tilde{R}_{25,16}(u,v)\\
&= -\frac{(1+uv)(1+e^{2\eta}uv)(1+e^{4\eta}uv)^2\sqrt{(1+e^{2\eta}u^2)(1+e^{4\eta}v^2)}}{e^{\eta}(u-v)^2(e^{2\eta}u-v)(u-e^{2\eta}v)(u-e^{4\eta}v)},\\
\tilde{R}_{14,41}(u,v) &= \tilde{R}_{71,17}(u,v) = e^{2\eta}\tilde{R}_{16,61}(u,v) = e^{2\eta}\tilde{R}_{61,16}(u,v) = e^{4\eta}\tilde{R}_{17,71}(u,v) = e^{4\eta}\tilde{R}_{41,14}(u,v)\\
&= -\frac{(1+uv)(1+e^{2\eta}uv)(1+e^{4\eta}uv)^2\sqrt{(1+e^{2\eta}u^2)(1+e^{4\eta}u^2)(1+e^{2\eta}v^2)(1+e^{4\eta}v^2)}}{(u-v)^2(e^{2\eta}u-v)(u-e^{2\eta}v)(e^{4\eta}u-v)(u-e^{4\eta}v)},\\
\tilde{R}_{18,18}(u,v) &= \frac{(1+uv)(1+e^{2\eta}uv)(1+e^{4\eta}uv)}{(u-v)(u-e^{2\eta}v)(u-e^{4\eta}v)},\\
\tilde{R}_{18,27}(u,v) &= \tilde{R}_{54,18}(u,v) = e^{\eta}\tilde{R}_{18,36}(u,v) = e^{\eta}\tilde{R}_{36,18}(u,v) = e^{2\eta}\tilde{R}_{18,54}(u,v) = e^{2\eta}\tilde{R}_{27,18}(u,v)\\
&= \frac{(1+uv)(1+e^{2\eta}uv)(1+e^{4\eta}uv)\sqrt{(1+u^2)(1+e^{4\eta}v^2)}}{(u-v)^2(u-e^{2\eta}v)(u-e^{4\eta}v)},\\
\tilde{R}_{18,45}(u,v) &= \tilde{R}_{72,18}(u,v) = e^{\eta}\tilde{R}_{18,63}(u,v) = e^{\eta}\tilde{R}_{63,18}(u,v) = e^{2\eta}\tilde{R}_{18,72}(u,v) = e^{2\eta}\tilde{R}_{45,18}(u,v)\\
&= \frac{(1+uv)(1+e^{2\eta}uv)(1+e^{4\eta}uv)\sqrt{(1+u^2)(1+e^{2\eta}u^2)(1+e^{2\eta}v^2)(1+e^{4\eta}v^2)}}{(u-v)^2(e^{2\eta}u-v)(u-e^{2\eta}v)(u-e^{4\eta}v)},
\end{aligned}
$$

$$
\begin{aligned}
\tilde{R}_{18,81}(u,v) &= \tilde{R}_{81,18}(u,v) = \frac{(1+uv)(1+e^{2\eta}uv)(1+e^{4\eta}uv)}{(u-v)^2(e^{2\eta}u-v)(u-e^{2\eta}v)(e^{4\eta}u-v)(u-e^{4\eta}v)} \\
&\qquad \times\sqrt{(1+u^2)(1+e^{2\eta}u^2)(1+e^{4\eta}u^2)(1+v^2)(1+e^{2\eta}v^2)(1+e^{4\eta}v^2)}, \\
\tilde{R}_{21,21}(u,v) &= \tilde{R}_{31,31}(u,v) = \tilde{R}_{51,51}(u,v) = \frac{(1+uv)(1+e^{2\eta}uv)(1+e^{4\eta}uv)^2(1+e^{6\eta}uv)}{e^{4\eta}(u-v)^2(e^{2\eta}u-v)(u-e^{2\eta}v)(e^{4\eta}u-v)}, \\
\tilde{R}_{22,22}(u,v) &= \tilde{R}_{33,33}(u,v) = \tilde{R}_{55,55}(u,v) = -\frac{(1+uv)^2(1+e^{2\eta}uv)(1+e^{4\eta}uv)^2(1+e^{6\eta}uv)}{e^{2\eta}(u-v)^2(e^{2\eta}u-v)(u-e^{2\eta}v)(e^{4\eta}u-v)(u-e^{4\eta}v)}, \\
\tilde{R}_{23,23}(u,v) &= \tilde{R}_{25,25}(u,v) = \tilde{R}_{32,32}(u,v) = \tilde{R}_{35,35}(u,v) = \tilde{R}_{52,52}(u,v) = \tilde{R}_{53,53}(u,v) \\
&= -\frac{(1+uv)(1+e^{2\eta}uv)(1+e^{4\eta}uv)^2}{e^{3\eta}(u-v)^2(e^{2\eta}u-v)(u-e^{2\eta}v)}, \\
\tilde{R}_{23,32}(u,v) &= \tilde{R}_{35,53}(u,v) = e^{\eta}\tilde{R}_{25,52}(u,v) \\
&= -\frac{(1+uv)(1+e^{2\eta}uv)(1+e^{4\eta}uv)^2(e^{2\eta}(1+e^{2\eta}u^2)(1+e^{2\eta}v^2)+(1-e^{2\eta})(1+e^{6\eta}uv))}{e^{4\eta}(u-v)^2(e^{2\eta}u-v)(u-e^{2\eta}v)(e^{4\eta}u-v)(u-e^{4\eta}v)}, \\
\tilde{R}_{23,41}(u,v) &= \tilde{R}_{71,53}(u,v) = e^{-\eta}\tilde{R}_{32,41}(u,v) = e^{-\eta}\tilde{R}_{52,61}(u,v) \\
&= e^{-\eta}\tilde{R}_{61,25}(u,v) = e^{-\eta}\tilde{R}_{71,35}(u,v) = e^{\eta}\tilde{R}_{41,23}(u,v) = e^{\eta}\tilde{R}_{53,71}(u,v) \\
&= e^{2\eta}\tilde{R}_{25,61}(u,v) = e^{2\eta}\tilde{R}_{35,71}(u,v) = e^{2\eta}\tilde{R}_{41,32}(u,v) = e^{2\eta}\tilde{R}_{61,52}(u,v) \\
&= -\frac{(1+uv)(1+e^{2\eta}uv)(1+e^{4\eta}uv)^2\sqrt{(1+e^{4\eta}u^2)(1+e^{2\eta}v^2)}}{e^{2\eta}(u-v)^2(e^{2\eta}u-v)(u-e^{2\eta}v)(e^{4\eta}u-v)}, \\
\tilde{R}_{24,24}(u,v) &= \tilde{R}_{26,26}(u,v) = \tilde{R}_{34,34}(u,v) = \tilde{R}_{37,37}(u,v) = \tilde{R}_{56,56}(u,v) = \tilde{R}_{57,57}(u,v) \\
&= \frac{(1+uv)^2(1+e^{2\eta}uv)(1+e^{4\eta}uv)^2}{e^{\eta}(u-v)^2(e^{2\eta}u-v)(u-e^{2\eta}v)(u-e^{4\eta}v)}, \\
\tilde{R}_{24,42}(u,v) &= \tilde{R}_{42,24}(u,v) = \tilde{R}_{57,75}(u,v) = \tilde{R}_{75,57}(u,v) \\
&= e^{2\eta}\tilde{R}_{26,62}(u,v) = e^{2\eta}\tilde{R}_{37,73}(u,v) = e^{2\eta}\tilde{R}_{43,34}(u,v) = e^{2\eta}\tilde{R}_{65,56}(u,v) \\
&= e^{-2\eta}\tilde{R}_{34,43}(u,v) = e^{-2\eta}\tilde{R}_{56,65}(u,v) = e^{-2\eta}\tilde{R}_{62,26}(u,v) = e^{-2\eta}\tilde{R}_{73,37}(u,v) \\
&= \frac{(1+uv)^2(1+e^{2\eta}uv)(1+e^{4\eta}uv)^2\sqrt{(1+e^{2\eta}u^2)(1+e^{2\eta}v^2)}}{(u-v)^2(e^{2\eta}u-v)(u-e^{2\eta}v)(e^{4\eta}u-v)(u-e^{4\eta}v)}, \\
\tilde{R}_{27,27}(u,v) &= \tilde{R}_{36,36}(u,v) = \tilde{R}_{54,54}(u,v) = \frac{(1+uv)(1+e^{2\eta}uv)(1+e^{4\eta}uv)}{e^{2\eta}(u-v)^2(u-e^{2\eta}v)}, \\
\tilde{R}_{27,36}(u,v) &= \tilde{R}_{36,54}(u,v) = e^{\eta}\tilde{R}_{27,54}(u,v) \\
&= \frac{(1+uv)(1+e^{2\eta}uv)(1+e^{4\eta}uv)((1+uv)(1+e^{4\eta}uv)+(e^{2\eta}u-v)(u-e^{4\eta}v))}{e^{3\eta}(u-v)^2(e^{2\eta}u-v)(u-e^{2\eta}v)(u-e^{4\eta}v)}, \\
\tilde{R}_{27,45}(u,v) &= \tilde{R}_{45,27}(u,v) = \tilde{R}_{54,72}(u,v) = \tilde{R}_{72,54}(u,v) \\
&= e^{\eta}\tilde{R}_{27,63}(u,v) = e^{\eta}\tilde{R}_{36,72}(u,v) = e^{\eta}\tilde{R}_{45,36}(u,v) = e^{\eta}\tilde{R}_{63,54}(u,v) \\
&= e^{-\eta}\tilde{R}_{36,45}(u,v) = e^{-\eta}\tilde{R}_{54,63}(u,v) = e^{-\eta}\tilde{R}_{63,27}(u,v) = e^{-\eta}\tilde{R}_{72,36}(u,v) \\
&= \frac{(1+uv)(1+e^{2\eta}uv)(1+e^{4\eta}uv)\sqrt{(1+e^{2\eta}u^2)(1+e^{2\eta}v^2)}}{e^{2\eta}(u-v)^2(e^{2\eta}u-v)(u-e^{2\eta}v)}, \\
\tilde{R}_{27,72}(u,v) &= \tilde{R}_{45,54}(u,v) = \sqrt{(1+e^{2\eta}u^2)(1+e^{2\eta}v^2)} \\
&\qquad \times\frac{(1+uv)(1+e^{2\eta}uv)(1+e^{4\eta}uv)((1+e^{4\eta}u^2)(1+e^{4\eta}v^2)+e^{4\eta}uv(1-e^{4\eta})(1+uv))}{e^{4\eta}(u-v)^2(e^{2\eta}u-v)(u-e^{2\eta}v)(e^{4\eta}u-v)(u-e^{4\eta}v)}, \\
\tilde{R}_{27,81}(u,v) &= \tilde{R}_{81,54}(u,v) = e^{-\eta}\tilde{R}_{36,81}(u,v) = e^{-\eta}\tilde{R}_{81,36}(u,v) = e^{-2\eta}\tilde{R}_{54,81}(u,v) = e^{-2\eta}\tilde{R}_{81,27}(u,v) \\
&= \frac{(1+uv)(1+e^{2\eta}uv)(1+e^{4\eta}uv)\sqrt{(1+e^{2\eta}u^2)(1+e^{4\eta}u^2)(1+v^2)(1+e^{2\eta}v^2)}}{e^{2\eta}(u-v)^2(e^{2\eta}u-v)(u-e^{2\eta}v)(e^{4\eta}u-v)}, \\
\tilde{R}_{28,28}(u,v) &= \tilde{R}_{38,38}(u,v) = \tilde{R}_{58,58}(u,v) = -\frac{(1+uv)^2(1+e^{2\eta}uv)(1+e^{4\eta}uv)}{(u-v)^2(u-e^{2\eta}v)(u-e^{4\eta}v)},
\end{aligned}
$$

$$
\begin{aligned}
\tilde{R}_{28,46}(u,v) &= \tilde{R}_{76,58}(u,v) = e^{\eta}\tilde{R}_{28,64}(u,v) = e^{\eta}\tilde{R}_{38,74}(u,v) \\
&= e^{\eta}\tilde{R}_{47,38}(u,v) = e^{\eta}\tilde{R}_{67,58}(u,v) = e^{-\eta}\tilde{R}_{46,28}(u,v) = e^{-\eta}\tilde{R}_{58,76}(u,v) \\
&= e^{-2\eta}\tilde{R}_{38,47}(u,v) = e^{-2\eta}\tilde{R}_{58,67}(u,v) = e^{-2\eta}\tilde{R}_{64,28}(u,v) = e^{-2\eta}\tilde{R}_{74,38}(u,v) \\
&= -\frac{(1+uv)^2(1+e^{2\eta}uv)(1+e^{4\eta}uv)\sqrt{(1+u^2)(1+e^{2\eta}v^2)}}{(u-v)^2(u-e^{2\eta}v)(e^{2\eta}u-v)(u-e^{4\eta}v)}, \\
\tilde{R}_{28,82}(u,v) &= \tilde{R}_{85,58}(u,v) = e^{-2\eta}\tilde{R}_{38,83}(u,v) = e^{-2\eta}\tilde{R}_{83,38}(u,v) = e^{-4\eta}\tilde{R}_{58,85}(u,v) = e^{-4\eta}\tilde{R}_{82,28}(u,v) \\
&= -\frac{(1+uv)^2(1+e^{2\eta}uv)(1+e^{4\eta}uv)\sqrt{(1+u^2)(1+e^{2\eta}u^2)(1+v^2)(1+e^{2\eta}v^2)}}{(u-v)^2(e^{2\eta}u-v)(u-e^{2\eta}v)(e^{4\eta}u-v)(u-e^{4\eta}v)}, \\
\tilde{R}_{32,23}(u,v) &= \tilde{R}_{53,35}(u,v) = e^{-2\eta}\tilde{R}_{52,25}(u,v) \\
&= -\frac{(1+uv)(1+e^{2\eta}uv)(1+e^{4\eta}uv)^2(e^{2\eta}(1+e^{2\eta}u^2)(1+e^{2\eta}v^2)+uv(e^{2\eta}-1)(1+e^{6\eta}uv))}{e^{2\eta}(u-v)^2(e^{2\eta}u-v)(u-e^{2\eta}v)(e^{4\eta}u-v)(u-e^{4\eta}v)}, \\
\tilde{R}_{36,27}(u,v) &= \tilde{R}_{54,36}(u,v) = e^{-\eta}\tilde{R}_{54,27}(u,v) \\
&= \frac{(1+uv)(1+e^{2\eta}uv)(1+e^{4\eta}uv)(e^{2\eta}(1+uv)(1+e^{4\eta}uv)+(e^{2\eta}u-v)(u-e^{4\eta}v))}{e^{\eta}(u-v)^2(e^{2\eta}u-v)(u-e^{2\eta}v)(u-e^{4\eta}v)}, \\
\tilde{R}_{36,63}(u,v) &= \tilde{R}_{63,36}(u,v) = \sqrt{(1+e^{2\eta}u^2)(1+e^{2\eta}v^2)} \\
&\times\frac{(1+uv)(1+e^{2\eta}uv)(1+e^{4\eta}uv)[e^{2\eta}(1+e^{2\eta}u^2)(1+e^{2\eta}v^2)-uv(1-e^{2\eta})(1-e^{6\eta})]}{e^{2\eta}(u-v)^2(e^{2\eta}u-v)(u-e^{2\eta}v)(e^{4\eta}u-v)(u-e^{4\eta}v)}, \\
\tilde{R}_{41,41}(u,v) &= \tilde{R}_{61,61}(u,v) = \tilde{R}_{71,71}(u,v) = -\frac{(1+uv)(1+e^{2\eta}uv)(1+e^{4\eta}uv)^2}{e^{2\eta}(u-v)^2(e^{2\eta}u-v)(e^{4\eta}u-v)}, \\
\tilde{R}_{42,42}(u,v) &= \tilde{R}_{43,43}(u,v) = \tilde{R}_{62,62}(u,v) = \tilde{R}_{65,65}(u,v) = \tilde{R}_{73,73}(u,v) = \tilde{R}_{75,75}(u,v) \\
&= \frac{(1+uv)^2(1+e^{2\eta}uv)(1+e^{4\eta}uv)^2}{e^{\eta}(u-v)^2(e^{2\eta}u-v)(u-e^{2\eta}v)(e^{4\eta}u-v)}, \\
\tilde{R}_{44,44}(u,v) &= \tilde{R}_{66,66}(u,v) = \tilde{R}_{77,77}(u,v) \\
&= -\frac{(1+uv)^2(1+e^{2\eta}uv)(1+e^{4\eta}uv)^2(uv+e^{2\eta})}{(u-v)^2(e^{2\eta}u-v)(u-e^{2\eta}v)(e^{4\eta}u-v)(u-e^{4\eta}v)}, \\
\tilde{R}_{45,45}(u,v) &= \tilde{R}_{63,63}(u,v) = \tilde{R}_{72,72}(u,v) = \frac{(1+uv)(1+e^{2\eta}uv)(1+e^{4\eta}uv)}{e^{2\eta}(u-v)^2(e^{2\eta}u-v)}, \\
\tilde{R}_{45,63}(u,v) &= \tilde{R}_{63,72}(u,v) = e^{\eta}\tilde{R}_{45,72}(u,v) \\
&= \frac{(1+uv)(1+e^{2\eta}uv)(1+e^{4\eta}uv)((1+uv)(1+e^{4\eta}uv)+(e^{4\eta}u-v)(u-e^{2\eta}v))}{e^{3\eta}(u-v)^2(e^{2\eta}u-v)(u-e^{2\eta}v)(e^{4\eta}u-v)}, \\
\tilde{R}_{45,81}(u,v) &= \tilde{R}_{81,72}(u,v) = e^{-\eta}\tilde{R}_{63,81}(u,v) = e^{-\eta}\tilde{R}_{81,63}(u,v) = e^{-2\eta}\tilde{R}_{72,81}(u,v) = e^{-2\eta}\tilde{R}_{81,45}(u,v) \\
&= \frac{(1+uv)(1+e^{2\eta}uv)(1+e^{4\eta}uv)\sqrt{(1+v^2)(1+e^{4\eta}u^2)}}{e^{2\eta}(u-v)^2(e^{2\eta}u-v)(e^{4\eta}u-v)}, \\
\tilde{R}_{46,46}(u,v) &= \tilde{R}_{47,47}(u,v) = \tilde{R}_{64,64}(u,v) = \tilde{R}_{67,67}(u,v) = \tilde{R}_{74,74}(u,v) = \tilde{R}_{76,76}(u,v) \\
&= -\frac{(1+uv)^2(1+e^{2\eta}uv)(1+e^{4\eta}uv)}{e^{\eta}(u-v)^2(e^{2\eta}u-v)(u-e^{2\eta}v)}, \\
\tilde{R}_{46,64}(u,v) &= \tilde{R}_{47,74}(u,v) = \tilde{R}_{67,76}(u,v) \\
&= -\frac{(1+uv)^2(1+e^{2\eta}uv)(1+e^{4\eta}uv)((1+e^{2\eta}u^2)(1+e^{2\eta}v^2)+uve^{2\eta}(1-e^{2\eta})(uv+e^{2\eta}))}{(u-v)^2(e^{2\eta}u-v)(u-e^{2\eta}v)(e^{4\eta}u-v)(u-e^{4\eta}v)}, \\
\tilde{R}_{46,82}(u,v) &= \tilde{R}_{47,83}(u,v) = \tilde{R}_{83,74}(u,v) = \tilde{R}_{85,76}(u,v) \\
&= e^{-\eta}\tilde{R}_{64,82}(u,v) = e^{-\eta}\tilde{R}_{85,67}(u,v) = e^{-2\eta}\tilde{R}_{67,85}(u,v) = e^{-2\eta}\tilde{R}_{82,64}(u,v) \\
&= e^{-3\eta}\tilde{R}_{74,83}(u,v) = e^{-3\eta}\tilde{R}_{76,85}(u,v) = e^{-3\eta}\tilde{R}_{82,46}(u,v) = e^{-3\eta}\tilde{R}_{83,47}(u,v) \\
&= -\frac{(1+uv)^2(1+e^{2\eta}uv)(1+e^{4\eta}uv)\sqrt{(1+v^2)(1+e^{2\eta}u^2)}}{e^{\eta}(u-v)^2(e^{2\eta}u-v)(u-e^{2\eta}v)(e^{4\eta}u-v)},
\end{aligned}
$$

$$\tilde{R}_{48,48}(u,v) = \tilde{R}_{68,68}(u,v) = \tilde{R}_{78,78}(u,v) = \frac{(1+uv)^2(1+e^{2\eta}uv)(1+e^{4\eta}uv)(uv+e^{2\eta})}{(u-v)^2(e^{2\eta}u-v)(u-e^{2\eta}v)(u-e^{4\eta}v)},$$

$$\tilde{R}_{48,84}(u,v) = \tilde{R}_{87,78}(u,v) = e^{-2\eta}\tilde{R}_{68,86}(u,v) = e^{-2\eta}\tilde{R}_{86,68}(u,v) = e^{-4\eta}\tilde{R}_{78,87}(u,v) = e^{-4\eta}\tilde{R}_{84,48}(u,v) = \frac{(1+uv)^2(1+e^{2\eta}uv)(1+e^{4\eta}uv)(uv+e^{2\eta})\sqrt{(1+u^2)(1+v^2)}}{(u-v)^2(e^{2\eta}u-v)(u-e^{2\eta}v)(e^{4\eta}u-v)(u-e^{4\eta}v)},$$

$$\tilde{R}_{54,45}(u,v) = \tilde{R}_{72,27}(u,v) = \sqrt{(1+e^{2\eta}u^2)(1+e^{2\eta}v^2)} \times \frac{(1+uv)(1+e^{2\eta}uv)(1+e^{4\eta}uv)((1+e^{4\eta}u^2)(1+e^{4\eta}v^2)+(e^{4\eta}-1)(1+uv))}{(u-v)^2(e^{2\eta}u-v)(u-e^{2\eta}v)(e^{4\eta}u-v)(u-e^{4\eta}v)},$$

$$\tilde{R}_{63,45}(u,v) = \tilde{R}_{72,63}(u,v) = e^{-\eta}\tilde{R}_{72,45}(u,v) = \frac{(1+uv)(1+e^{2\eta}uv)(1+e^{4\eta}uv)(e^{2\eta}(1+uv)(1+e^{4\eta}uv)+(e^{4\eta}u-v)(u-e^{2\eta}v))}{e^{\eta}(u-v)^2(e^{2\eta}u-v)(u-e^{2\eta}v)(e^{4\eta}u-v)},$$

$$\tilde{R}_{64,46}(u,v) = \tilde{R}_{76,67}(u,v) = e^{-2\eta}\tilde{R}_{74,47}(u,v) = -\frac{(1+uv)^2(1+e^{2\eta}uv)(1+e^{4\eta}uv)(e^{2\eta}(1+e^{2\eta}u^2)(1+e^{2\eta}v^2)+(e^{2\eta}-1)(uv+e^{2\eta}))}{(u-v)^2(e^{2\eta}u-v)(u-e^{2\eta}v)(e^{4\eta}u-v)(u-e^{4\eta}v)},$$

$$\tilde{R}_{81,81}(u,v) = \frac{(1+uv)(1+e^{2\eta}uv)(1+e^{4\eta}uv)}{(u-v)(e^{2\eta}u-v)(e^{4\eta}u-v)},$$

$$\tilde{R}_{82,82}(u,v) = \tilde{R}_{83,83}(u,v) = \tilde{R}_{85,85}(u,v) = -\frac{(1+uv)^2(1+e^{2\eta}uv)(1+e^{4\eta}uv)}{(u-v)^2(e^{2\eta}u-v)(e^{4\eta}u-v)},$$

$$\tilde{R}_{84,48}(u,v) = \tilde{R}_{86,68}(u,v) = \tilde{R}_{87,78}(u,v) = \frac{(1+uv)^2(1+e^{2\eta}uv)(1+e^{4\eta}uv)(uv+e^{2\eta})}{(u-v)^2(e^{2\eta}u-v)(u-e^{2\eta}v)(e^{4\eta}u-v)},$$

$$\tilde{R}_{88,88}(u,v) = -\frac{(1+uv)^2(1+e^{2\eta}uv)(1+e^{4\eta}uv)(uv+e^{2\eta})(uv+e^{4\eta})}{(u-v)^2(e^{2\eta}u-v)(u-e^{2\eta}v)(e^{4\eta}u-v)(u-e^{4\eta}v)}$$

Having found the matrix $\tilde{R}(u,v)$, we can proceed to solve the RE for $K_+(u)$. The procedure is similar to that for $K_-(u)$, as sketched in the previous Appendix. The final result is given by (III.7).

APPENDIX C: ON CROSSING-UNITARITY

As a sideline, we show in this appendix that all R-matrices associated with finite-dimensional representations of the quantum affine algebra $U_q[\mathcal{G}^{(k)}]$ $(k=1,2)$ for generic q, where $\mathcal{G}$ is any bosonic Lie algebra, enjoy the crossing unitarity property. We address, however, that the R-matrix appeared in the present paper does not belong to this class.

It is shown in [12] that for any pair of finite dimensional $U_q[\mathcal{G}^{(1)}]$-modules V and W, the R-matrix $R^{VW}(z)$ satisfies the crossing-unitarity relations:

$$\begin{aligned}(((R^{VW}(z)^{-1})^{t_1})^{-1})^{t_1} &= (\pi_V(q^{2\rho})\otimes 1_W)R^{VW}(zq^{2g})(\pi_V(q^{-2\rho})\otimes 1_W),\\ (((R^{VW}(z)^{-1})^{t_2})^{-1})^{t_2} &= (1_V\otimes \pi_W(q^{-2\rho}))R^{VW}(zq^{-2g})(1_V\otimes \pi_W(q^{2\rho})).\end{aligned} \tag{C.1}$$

where ρ is the half-sum of the positive roots of $\mathcal{G}$, and $2g=(\psi,\psi+2\rho)$, with ψ being the highest root of $\mathcal{G}$.

We now extend the arguments of [12] to R-matrices associated with the twisted quantum affine bosonic algebra $U_q[\mathcal{G}^{(2)}]$, and prove the similar crossing-unitarity relations.

Let us first of all recall some facts about the twisted affine algebra $\mathcal{G}^{(2)}$. Let $\mathcal{G}_0$ be the fixed point subalgebra under the diagram automorphism $\hat{\tau}$ of $\mathcal{G}$ of order 2. We can decompose $\mathcal{G}$ as $\mathcal{G}_0$ and a $\mathcal{G}_0$-representation $\mathcal{G}_1$ of $\mathcal{G}$. Let θ_0 be the highest weight of the $\mathcal{G}_0$-representation $\mathcal{G}_1$. Following the usual convention, we denote the weight of $\mathcal{G}^{(2)}$ by $\Lambda\equiv(\lambda,\kappa,\tau)$, where λ is a weight of $\mathcal{G}_0$. With this notation the nondegenerate form (,) induced on the weights can be expressed as

$$(\Lambda,\Lambda') = (\lambda,\lambda') + \kappa\tau' + \kappa'\tau. \tag{C.2}$$

The fundamental dominant weights of $\mathcal{G}^{(2)}$, $\lambda_{(i)}$ $(0\le i\le r)$, can be shown to be

$$\lambda_{(0)} = (\theta_0, \theta_0)\gamma, \quad \lambda_{(i)} = (\lambda_i, 0, 0) + 2(\lambda_i, \theta_0)\gamma, \quad i = 1, \cdots, r \tag{C.3}$$

where λ_i are fundamental dominant weights of $\mathcal{G}_0$ and $\gamma = (0, 1, 0)$. The distinguished dominant weight is

$$\widehat{\rho} = \sum_{i=0}^{r} \lambda_{(i)} = \rho + g_0\gamma \tag{C.4}$$

where $g_0 = (\theta_0, \theta_0 + 2\rho)$ and $\rho\ (= \sum_{i=1}^{r} \lambda_i)$ is the half-sum of positive roots of $\mathcal{G}_0$.

We shall not give the defining relations for $U_q[\mathcal{G}^{(2)}]$, but mention that the actions of coproduct and antipode on its generators $\{h_i,\ e_i,\ f_i,\ d,\ 0 \le i \le r\}$ are given by

$$\begin{aligned}
&\Delta(h_i) = h_i \otimes 1 + 1 \otimes h_i, \quad \Delta(d) = d \otimes 1 + 1 \otimes d,\\
&\Delta(e_i) = e_i \otimes q^{\frac{h_i}{2}} + q^{-\frac{h_i}{2}} \otimes e_i, \quad \Delta(f_i) = f_i \otimes q^{\frac{h_i}{2}} + q^{-\frac{h_i}{2}} \otimes f_i,\\
&S(a) = -q^{\widehat{\rho}}\, a\, q^{-\widehat{\rho}}, \quad \forall a = d,\ h_i,\ e_i,\ f_i.
\end{aligned} \tag{C.5}$$

Define an automorphism D_z of $U_q[\mathcal{G}^{(2)}]$ by

$$D_z(e_i) = z^{\delta_{i0}} e_i, \quad D_z(f_i) = z^{-\delta_{i0}} f_i, \quad D_z(h_i) = h_i, \quad D_z(d) = d. \tag{C.6}$$

Then it can be shown that

$$S^2(a) = q^{2\rho}\, D_{q^{g_0}}\,(a)\, q^{-2\rho}, \qquad S^{-2}(a) = q^{-2\rho}\, D_{q^{-g_0}}\,(a)\, q^{2\rho} \tag{C.7}$$

Following [12], we define the right dual module V^* and left dual module ${}^*V$ by

$$\pi_{V^*}(a) = \pi_V(S(a))^t, \qquad \pi_{{}^*V}(a) = \pi_V(S^{-1}(a))^t, \tag{C.8}$$

respectively. By the same arguments as in [12], one can show that

$$R^{V^*,W}(z) = (R^{VW}(z)^{-1})^{t_1}, \qquad R^{V,{}^*W}(z) = (R^{VW}(z)^{-1})^{t_2}. \tag{C.9}$$

It follows from the representations of $R^{V^{**},W}(z)$ and $R^{V,{}^{**}W}(z)$ that for any pair of finite dimensional $U_q[\mathcal{G}^{(2)}]$-modules V and W, the R-matrix satisfies the following crossing-unitarity relations

$$\begin{aligned}
&(((R^{VW}(z)^{-1})^{t_1})^{-1})^{t_1} = (\pi_V(q^{2\rho}) \otimes 1_W) R^{VW}(zq^{g_0})(\pi_V(q^{-2\rho}) \otimes 1_W),\\
&(((R^{VW}(z)^{-1})^{t_2})^{-1})^{t_2} = (1_V \otimes \pi_W(q^{-2\rho})) R^{VW}(zq^{-g_0})(1_V \otimes \pi_W(q^{2\rho})).
\end{aligned} \tag{C.10}$$

Note also that

$$(\pi_V(q^{\pm 2\rho}) \otimes \pi_W(q^{\pm 2\rho})) R^{VW}(z) = R^{VW}(z)(\pi_V(q^{\pm 2\rho}) \otimes \pi_W(q^{\pm 2\rho})). \tag{C.11}$$

Let us remark that if one uses the opposite coproduct and antipode of $U_q[\mathcal{G}^{(2}]$,

$$\begin{aligned}
&\bar{\Delta}(h_i) = h_i \otimes 1 + 1 \otimes h_i, \quad \bar{\Delta}(d) = d \otimes 1 + 1 \otimes d,\\
&\bar{\Delta}(e_i) = e_i \otimes q^{-\frac{h_i}{2}} + q^{\frac{h_i}{2}} \otimes e_i, \quad \bar{\Delta}(f_i) = f_i \otimes q^{-\frac{h_i}{2}} + q^{\frac{h_i}{2}} \otimes f_i,\\
&\bar{S}(a) = -q^{-\widehat{\rho}}\, a\, q^{\widehat{\rho}}, \quad \forall a = d,\ h_i,\ e_i,\ f_i,
\end{aligned} \tag{C.12}$$

and denote the corresponding R-matrix by $\bar{R}(z)$, then the similar arguments as above give rise to the following crossing-untarity relations:

$$\begin{aligned}
&(((\bar{R}^{VW}(z)^{-1})^{t_1})^{-1})^{t_1} = (\pi_V(q^{-2\rho}) \otimes 1_W) \bar{R}^{VW}(zq^{-g_0})(\pi_V(q^{2\rho}) \otimes 1_W),\\
&(((\bar{R}^{VW}(z)^{-1})^{t_2})^{-1})^{t_2} = (1_V \otimes \pi_W(q^{2\rho})) \bar{R}^{VW}(zq^{g_0})(1_V \otimes \pi_W(q^{-2\rho})).
\end{aligned} \tag{C.13}$$

---

[1] E.K. Sklyanin, J. Phys. **A:** Math.Gen. **21** (1988) 2375.

[2] L. Mezincescu, R. Nepomechie, J. Phys. **A:** Math. Gen. **24** (1991) L17; Int. J. Mod. Phys. **A6** (1991) 5231.
[3] H.J. de Vega, A. González-Ruiz, J. Phys. **A:** Math. Gen. **26** (1993) L519; Mod. Phys. Lett. **A9** (1994) 2207; J. Abad, M. Rios, Phys. Lett. **B352** (1995) 92.
[4] C.M. Yung, M.T. Batchelor, Nucl. Phys. **B435** (1995) 430; M. Jimbo, R. Kedem, T. Kojima, H. Konno, T. Miwa, Nucl. Phys. **441** (1995) 437.
[5] A. González-Ruiz, Nucl. Phys. **B424** (1994) 553; H.-Q. Zhou, Phys. Rev. **B54** (1996) 41.
[6] H. Asakawa, M. Suzuki, J. Phys. **A:** Math. Gen. **29** (1996) 225; M. Shiroishi, M. Wadati, J. Phys. Soc. Jpn. **66** (1997) 1.
[7] H.-Q. Zhou, Phys. Rev. **B53** (1996) 5089.
[8] Y.-Z. Zhang, H.-Q. Zhou, preprints cond-mat/9707263; cond-mat/9708003.
[9] A.J. Bracken, X.-Y. Ge, Y.-Z. Zhang, H.-Q. Zhou, preprint cond-mat/9710141.
[10] H.-Q. Zhou, J. Phys. **A:** Math. Gen. **29** (1996) 5509; Phys. Lett. **A221** (1996) 104; H.-Q. Zhou, D.-M. Tong, Phys. Lett. **A232** (1997) 377.
[11] R.Z. Bariev, J. Phys. **A:** Math. Gen. **24** (1991) L919.
[12] N.Yu. Reshetikhin, M.A. Semenov-Tian-Shansky, Lett. Math. Phys. **19** (1990) 133; I.B. Frenkel, N.Yu. Reshetikhin, Commun. Math. Phys. **146** (1992) 1.